AY 20 Fall 2010

Stellar Interiors
Equations of Stellar Structure

Reading: Carroll & Ostlie, Chapter 10 §10.3, §10.4

First three equations of stellar structure

- equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r \rho}{r^2}$$

- mass conservation equation

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

- energy conservation equation

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$

and $$P = P_{gas} + P_{radiation} = \frac{\rho k T}{\mu m_H} + \tfrac{1}{3} a T^4$$

Remaining questions

- What is mean molecular weight, μ?
- What determines ε = total energy released /gm/sec?
  - nuclear burning (fusion) can sustain observed luminosity of Sun for > 10^{10} years
  - Temperature required for fusion ~ 10^7 K (if quantum tunneling allowed) ≈ central temp of Sun
  - and $\varepsilon = \varepsilon_{nuclear} + \varepsilon_{gravity}$
- How is energy transported and how does that affect temperature structure?

3

WHY μ? = "mean molecular weight" = $\frac{\bar{m}}{m_H}$

$\bar{m}$ = average mass of particles of different masses

INCLUDES ELECTRONS

∴ DEPENDS ON COMPOSITION & IONIZATION STATES OF EACH SPECIES!!

SIMPLE CASES: COMPLETELY NEUTRAL OR COMPLETELY IONIZED GAS

COMPLETELY NEUTRAL

$$\bar{m}_n = \sum_j N_j m_j / \sum_j N_j$$

m_j, N_j = mass & total number of atoms of type j

$$\therefore \mu_n = \frac{\bar{m}_n}{m_H} = \frac{\sum_j N_j A_j}{\sum_j N_j} \quad \text{and} \quad A_j = m_j / m_H$$

COMPLETELY IONIZED

$$\therefore \mu_c = \frac{\sum_j N_j A_j}{\sum_j (N_j + N_j z_j)} = \frac{\sum_j N_j A_j}{\sum_j N_j (1+z_j)}$$

Total number of particles = total number of ionized type-j atoms PLUS total number of electrons freed from each type-j atom

FROM ABOVE, FOR NEUTRAL ATOMS: $\frac{1}{\bar{m}_n} = \frac{\sum_j N_j}{\sum_j N_j m_j}$, $\therefore \frac{1}{\mu_n m_H} = \frac{\text{total number of particle}}{\text{total mass of gas}}$

$$\therefore \frac{1}{\mu_n m_H} = \frac{\text{total \# particles (neutral)}}{\text{total mass of gas}}$$

RECALL mass fraction = $\dfrac{\text{total mass of species}}{\text{total mass of gas}}$

FOR STARS & SUN: X Y Z

mass fractions H He METALS

we've shown $\dfrac{1}{\mu_n m_H} = \dfrac{\sum N_j}{\sum_j N_j m_j} = \dfrac{\text{total number of particles}}{\text{total mass of gas}}$

$= \sum_j \dfrac{\#\text{ of } j \text{ type particles}}{\text{mass of } j \text{ type particles}} \cdot \dfrac{\text{mass of } j \text{ type particles}}{\text{total mass of gas}}$

$\therefore \dfrac{1}{\mu_n m_H} = \sum_j \dfrac{N_j}{N_j m_j} \cdot X_j = \sum_j \dfrac{N_j}{N_j A_j m_H} \cdot X_j = \sum_j \dfrac{1}{A_j m_j} \cdot X_j$

and $A_j \equiv m_j/m_H$

$\therefore \dfrac{1}{\mu_n} = \sum_j \dfrac{1}{A_j} X_j$

$\therefore$ for neutral gas $\dfrac{1}{\mu_n} = X + \dfrac{1}{4} Y + \left\langle \dfrac{1}{A} \right\rangle_n Z$

for Sun $\left\langle \frac{1}{A} \right\rangle_n = 1/15.5$ = weighted average of all metals

FOR SUN $X = 0.70$ $Y = 0.28$ $Z = 0.02$

$\therefore \dfrac{1}{\mu_n} = 0.70 + 0.07 + 0.001 \ldots$

$\therefore \dfrac{1}{\mu_n} \sim 0.771$ and $\mu_n = 1.30$

FOR A COMPLETELY IONIZED GAS:

RECALL $\mu_i = \sum_j N_j A_j \Big/ \sum_j N_j (1+Z_j)$

(Z_j = # FREE ELECTRONS FROM COMPLETE IONIZATION OF ATOMS TYPE j)

NEED TOTAL NUMBER OF PARTICLES

HYDROGEN: NUCLEUS + 1 FREE ELECTRON

HELIUM: NUCLEUS + 2 FREE ELECTRONS etc

BY ANALOGY WITH $\frac{1}{\mu_n} = \sum_j \frac{X_j}{A_j}$, $\frac{1}{\mu_i} = \sum_j \frac{1+Z_j}{A_j} X_j$

$$\therefore \frac{1}{\mu_i} \simeq 2X + \frac{3Y}{4} + \left\langle \frac{1+Z}{A} \right\rangle_i Z$$

FOR ELEMENTS MUCH HEAVIER THAN HELIUM

$1 + Z_j \simeq Z_j$ $\quad Z_j >> 1$ represents # electrons in atom type j

and $A \equiv m_j / m_H \simeq 2 Z_j$ since masses protons neutrons about equal

$$\therefore \left\langle \frac{1+Z}{A} \right\rangle_i \sim \frac{1}{2}$$

∴ for Sun (many low mass stars)

$$\frac{1}{\mu_i} = 2 \times 0.70 + \frac{3}{4} \times 0.28 + \frac{1}{2} 0.02 = 1.40 + 0.21 + 0.01$$

$\therefore \mu_i = 0.62$ mean molecular weight for ionized stars

recall $\mu_n = 1.30$

THE TRANSPORT OF ENERGY

6.

CONDUCTION

TRANSPORTS HEAT BY COLLISIONS BETWEEN PARTICLES [ELECTRONS]

BUT IN NORMAL STARS, FREQUENT COLLISIONS & NOT MUCH TRANSPORT. NOT EFFICIENT

RADIATIVE ENERGY TRANSPORT

ENERGY FROM NUCLEAR REACTIONS (& GRAVITION) CARRIED TO SURFACE BY PHOTONS

RECALL: PHOTONS ABSORBED, SCATTERED – OR RE-EMITTED – IN RANDOM DIRECTIONS AS THEY MOVE [NET FLOW] TO SURFACE.

OPACITY OF MATERIAL – NUMBER OF MEAN FREE PATHS TO SURFACE – IMPORTANT

– OBSTRUCTS/DIMINISHES I_λ

CONVECTION

HOT BUOYANT MASS ELEMENTS MOVE OUT

COOLER ELEMENTS FALL IN

Radiative Transport

RECALL: NET FLOW/FLUX OF PHOTONS TO REGIONS OF LOWER PRESSURE

$$P_{rad} = \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\,d\lambda = \frac{4\sigma T^4}{3c} = \frac{1}{3}aT^4$$

$\therefore$ AS T DECREASES, P DECREASES AWAY FROM CENTER OF STAR

$\rightarrow$ NET FLUX OUTWARD

where pressure gradient given by $\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c}F_{rad}$

from above $\frac{dP_{rad}}{dr} = \frac{4}{3}aT^3\frac{dT}{dr}$

$$\therefore \frac{4}{3}aT^3\frac{dT}{dr} = -\frac{\bar{\kappa}\rho}{c}F_{rad}$$

$$\therefore \frac{dT}{dr} = -\frac{3\bar{\kappa}\rho}{4ac}\cdot\frac{F_{rad}}{T^3}$$

For a star of radius r; $F_{rad} = \frac{L_r}{4\pi r^2}$

$$\therefore \frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$ = TEMPERATURE GRADIENT FOR RADIATIVE TRANSFER

Note: $-$ve sign: $\frac{dT}{dr}$ increases as r decreases – i.e. inwards (steepens)

temperature gradient also steepens if either opacity or density increases (or T decreases) $\equiv$ harder to transport energy

AT VERY HIGH $\frac{dT}{dr}$, RADIATION TRANSPORT INEFFICIENT

CONVECTION BECOMES IMPORTANT

– GAS MOTIONS TRANSMIT ENERGY

hot gases rise to cooler layers – lose energy – SINK

⇒ homogeneous material in convective zone

DIFFICULT TO MODEL 1) REQUIRES 3-d NAVIER-STOKES EQUATIONS OF FLUID MECHANICS

2) CHARACTERISTIC LENGTH SCALE ≡ PRESSURE SCALE HEIGHT

AND $H_P \approx$ STELLAR RADIUS (ALMOST)

3) TIMESCALE FOR FLUID ELEMENT TO TRAVERSE H_P ~ TIMESCALE FOR CHANGES IN STELLAR STRUCTURE

4) CONVECTION ≋ TURBULENCE (viscosity, heat dissipation)

→ APPROXIMATIONS AND 1-d CODES (not too bad)

DEFINE $\frac{1}{H_P} \equiv -\frac{1}{P}\frac{dP}{dr}$ $\therefore$ $P = P_0 e^{-r/H_P}$

PRESSURE SCALE HEIGHT

$\therefore$ for $r = H_P$, $P = P_0 e^{-1}$, H_P = distance over which P decreases by factor e

Since $\underbrace{\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}}_{\text{equn of hydrostatic equilibrium}} = -\rho g$; $\boxed{H_P = P/\rho g}$

local gravity

Calculate H_P for Sun assuming

$\bar{P} = P_{central}/2$, $\bar{\rho}_\odot$ average density, $R = R_\odot/2$

$$\therefore g = \frac{GM_r}{r^2} = \frac{6.7\times10^{-8}\times2\times10^{33}\times 1/2}{49\times10^{20}\times 1/4}$$

$$= \frac{24.8}{49}\times10^{5} \sim \frac{6.7}{12}\times10^{5}$$

$$g = 5.5\times10^{4}\ \mathrm{cm\ s^{-1}}$$

$$\therefore H_P = \frac{2.5\times10^{15}}{2}\times\left[\frac{2\times10^{33}}{4/3\,\pi\,(7\times10^{10})^{3}}\right]^{-1}\times\frac{1}{5.5\times10^{4}}$$

$$\frac{\cancel{2}\times10^{33}\times\cancel{3}}{2\ \cancel{4\pi}\times49\times7\times10^{30}} \sim \frac{10^{3}}{7\times10^{2}} \sim 1.4$$

$$\therefore H_P = \frac{2.5\times10^{15}}{11\times10^{4}}\times\frac{1}{1.4}\ \mathrm{cm}$$

$$= \frac{2.5}{15.4}\times10^{11}\times\frac{1}{7\times10^{10}}\ R_\odot$$

$$\sim R_\odot/4$$

$H_P \sim R_\odot/10$ from models

i.e. H_P and $R_\odot$ comparable magnitude

CONVECTIVE TRANSPORT

FROM THERMODYNAMICS :- SPECIFIC HEAT

$\equiv$ AMOUNT OF HEAT REQUIRED TO RAISE TEMPERATURE OF UNIT MASS OF MATERIAL BY UNIT TEMP INTERVAL

$$C_P \equiv \partial Q/\partial T\big|_P \qquad C_V = \partial Q/dT\big|_V$$

C_P, C_V specific heats at const pressure, const vol

∂Q amount of heat added at constant pressure or volume

RATIO OF SPECIFIC HEATS $\gamma = C_P/C_V$

FOR A MONATOMIC GAS $\gamma = 5/3$

CONSIDER GAS BUBBLE RISING (ADIABATICALLY EXPANDING)

- NO HEAT EXCHANGE WITH SURROUNDINGS

(at some point thermalizes → loses all heat)

FROM IDEAL GAS LAW: $P_g = \dfrac{\rho k T}{\mu m_H}$

$$\therefore \frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

Adiabatic $\therefore$ $P = K\rho^{\gamma}$ $\quad \therefore \dfrac{dP}{dr} = \dfrac{\gamma P}{\rho}\dfrac{d\rho}{dr}$

$\therefore$ for constant μ: $\dfrac{dP}{dr} = \dfrac{\cancel{\gamma} P}{\cancel{\rho} P}\dfrac{dP}{dr}\dfrac{\cancel{\rho}}{\gamma} + \dfrac{P}{T}\dfrac{dT}{dr}$

$$\therefore \left.\frac{dT}{dr}\right|_{adiabatic} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

→ how bubble's temperature changes with distance

$$\left.\frac{dT}{dr}\right|_{adiabatic} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

AND $P = \dfrac{\rho k T}{\mu m_H}$, $\dfrac{dM_r}{dr} = 4\pi r^2 \rho$

$$dP/dr = -GM_r\rho/r^2$$

$$\therefore \left.\frac{dT}{dr}\right|_{ad} = -\left(1-1/\gamma\right)\frac{\mu m_H}{\rho k}\frac{GM_r\rho}{r^2}$$

$$\therefore \left.\frac{dT}{dr}\right|_{adiabatic} = -\left(1-1/\gamma\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$$

AND $$\left.\frac{dT}{dr}\right|_{radiative} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L}{4\pi r^2}$$

COMPARISON OF TEMPERATURE GRADIENTS INDICATES CONDITIONS FOR RADIATION OR CONVECTIVE TRANSPORT

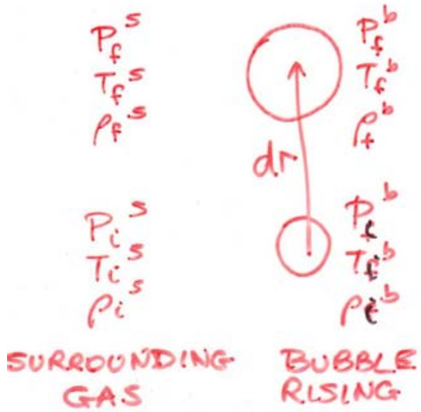
τ large $\rightarrow$ VERY STEEP $\left.\frac{dT}{dr}\right|_{rad}$ $\rightarrow$ CONVECTION (ATMOSPHERES)

ionization and large specific heat $\rightarrow$ low $\left.\frac{dT}{dr}\right|_{adiabatic}$ $\rightarrow$ CONVECTION (ATMOSPHERES)

energy generation conditions $\rightarrow$ steep $\left.\frac{dT}{dr}\right|_{rad}$ $\rightarrow$ CONVECTION (INTERIORS)

CONDITIONS FOR CONVECTION

– WHEN WILL HOT BUBBLE CONTINUE TO RISE RATHER THAN SINK?



i ≡ INITIAL CONDITIONS, f ≡ FINAL

BUBBLE WILL BEGIN TO RISE IF

$$\rho_i^b < \rho_i^s$$

FORCE ON BUBBLE

= BUOYANT FORCE/UNIT VOLUME – GRAVITIONAL FORCE

$\therefore f_{net} = \rho_i^s g - \rho_i^b g$ and $\delta\rho < 0$ to begin rise

AFTER DISTANCE dr, ρ_f^b AND ρ_f^s

$\therefore$ IF $\rho_f^b > \rho_f^s$ → NO CONVECTION, BUBBLE WILL SINK

IF $\rho_f^b < \rho_f^s$ → CONVECTION

HOW TO EXPRESS IN TERMS OF TEMPERATURE GRADIENTS?

ASSUME INITIALLY NEAR-THERMAL EQUILIBRIUM $T_i^b \simeq T_i^s$, $\rho_i^b \simeq \rho_i^s$

AND, AS BUBBLE EXPANDS ADIABATICALLY $P_f^b = P_f^s$ at all times

SINCE dr INFINITESIMAL, USE TAYLOR EXPANSION

$$\therefore \rho_f^b \simeq \rho_i^b + \frac{d\rho}{dr}\Big|^b dr \qquad \rho_f^s \simeq \rho_i^s + \frac{d\rho}{dr}\Big|^s dr$$

$\therefore$ CONVECTION CONDITION: $\rho_f^b < \rho_f^s \rightarrow \frac{d\rho}{dr}\Big|^b < \frac{d\rho}{dr}\Big|^s$

assuming almost constant densities inside & outside

WE HAVE: **CONVECTION CONDITION** $\left.\frac{d\rho}{dr}\right|^{b} < \left.\frac{d\rho}{dr}\right|^{s}$

WANT TO EXPRESS IN TERMS OF SURROUNDING CONDITIONS ONLY

adiabatic case $P = K\rho^{\gamma}$ ($PV^{\gamma} = K$)

RECALL ~~FROM LAST CLASS~~ : $\frac{dP}{dr} = \frac{\gamma P}{\rho}\frac{d\rho}{dr}$ (use for bubble)

(with μ constant)

AND $dP/dr = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$ with μ constant (use for surroundings)

$\therefore \left.\frac{d\rho}{dr}\right|^{b} = \frac{1}{\gamma}\frac{\rho^{b}}{P^{b}}\left.\frac{dP}{dr}\right|^{b} < \frac{\rho^{s}}{P_{s}}\left[\left.\frac{dP}{dr}\right|^{s} - \frac{P^{s}}{T^{s}}\left.\frac{dT}{dr}\right|^{s}\right]$

$\therefore$ since $\rho_i^{b} \sim \rho_i^{s}$ and $P^{b} = P^{s}$ at all times

$$\therefore \left.\frac{dP}{dr}\right|^{b} = \left.\frac{dP}{dr}\right|^{s} = \frac{dP}{dr}$$

$$\Rightarrow \frac{1}{\gamma}\frac{dP}{dr} < \frac{dP}{dr} - \frac{P^{s}}{T^{s}}\left.\frac{dT}{dr}\right|^{s}$$

$$\therefore \left(\frac{1}{\gamma} - 1\right)\frac{dP}{dr} < -\frac{P}{T}\frac{dT}{dr}_{actual}$$

actual temp gradient of surrounding gas

$$\therefore \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} > \frac{dT}{dr}_{actual} \quad \text{for convection}$$

LAST CLASS: $\left.\frac{dT}{dr}\right|_{ad} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$

(recall dP/dr also $-$ve)

??. **Convection Condition** ??

$$\left.\frac{dT}{dr}\right|_{ad} > \frac{dT}{dr}_{actual}$$