AY 20 Fall 2010

Stellar Interiors
Equations of Stellar Structure

Reading: Carroll & Ostlie, Chapter 10 §10.1 - §10.4

Stellar Structure Equations

These describe the structure of stellar interiors, assuming stars in equilibrium

Validity can be tested by
- observed parameters should match those computed from models based on structure equations

The equations govern:

- the variation in pressure with radius in the stellar interior
(equation of hydrostatic equilibrium)
- the distribution of mass
(equation of continuity - or mass conservation)
- the production of energy
(energy conservation equation)
- the transport of energy
(variation of temperature as a function of radius; depends on way energy is transported – by radiation, convection, or conduction)

First three equations

- equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

- mass conservation equation

$$\frac{dL}{dr} = 4\pi r^2 \rho\varepsilon$$

- energy conservation equation

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

and $$P = P_{gas} + P_{radiation} = \frac{\rho kT}{\mu m_H} + \tfrac{1}{3} a T^4$$

HYDROSTATIC EQUILIBRIUM IN STARS

GRAVITY = CONSTANT PULL TO COLLAPSE

MUST BE BALANCED BY:

PRESSURE – FROM THERMAL MOTIONS OF GAS

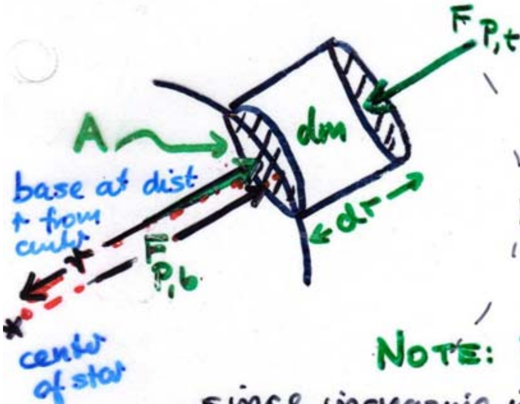
[CAN VARY WITH DISTANCE FROM STAR CENTER]

CONSIDER GRAVITATIONAL & PRESSURE FORCES ACTING ON A VOLUME ELEMENT, MASS dm

CYLINDER OF HEIGHT dr, BASE AREA A

$F_{P,t}$ = PRESSURE FORCE AT TOP

$F_{P,b}$ = PRESSURE FORCE AT BOTTOM



RECALL NEWTON'S 2^{ND}: $\underline{F} = m\underline{a}$

$$\therefore dm \frac{d^2r}{dt^2} = F_g + F_{P,t} + F_{P,b} = \text{net force}$$

NOTE: $F_g < 0$, so $F_g = -\frac{GM_r\,dm}{r^2}$

since increasing inwards

where M_r = mass interior to r = INTERIOR MASS

$$F_{P,t} < 0 \quad ; \quad F_{P,t} = -(F_{P,b} + dF_P)$$

since r different for top & bottom of cylinder

$$\therefore dm \frac{d^2r}{dt^2} = -\frac{GM_r\,dm}{r^2} - dF_P$$

$$dm\, d^2r/dt^2 = -\frac{GM_r\, dm}{r^2} - dF_p$$

SINCE PRESSURE = FORCE / UNIT AREA = F/A,

$$dF_p = A\,dP$$

AND DENSITY ρ IN CYLINDER OF

mass, $dm = \rho A\, dr$

SUBSTITUTING FOR dm and dF_p,

$$\therefore \rho A\, dr\, d^2r/dt^2 = -\frac{GM_r \rho A\, dr}{r^2} - A\,dP$$

$$\therefore \rho\, d^2r/dt^2 = -GM_r\rho/r^2 - dP/dr$$

IF ELEMENT IS STATIC $\Rightarrow$ FORCES BALANCE

AND $d^2r/dt^2 = 0$

$$\therefore dP/dr = -\frac{GM_r}{r^2}\rho = -\rho g;$$

where $g = -\frac{GM_r}{r^2}$ is local accel^n of gravity at radius r

EQUATION OF HYDROSTATIC EQUILIBRIUM : $\boxed{dP/dr = -\frac{GM_r\rho}{r^2}}$

N.B. GRAVITY BALANCED BY PRESSURE GRADIENT

(RECALL: STELLAR ATMOSPHERES: PRESSURE GRADIENT REGULATES FLUX; dP_{rad}/dr)

NOTE TOO: -ve SIGN $\therefore$ pressure greatest at center

$\boxed{\text{CHANGE IN PRESSURE}}$ WITH RADIUS

SUPPORTS STAR AGAINST GRAVITY

(spherically symm object)

Taking Sun as example: $M_* = 1\,M_\odot$, $r = 1\,R_\odot$

$$\rho = \bar{\rho}_\odot = 1410 \text{ kg m}^{-3} \quad (1.4 \text{ gm cm}^{-3})$$

ASSUME PRESSURE AT SURFACE IS 0

$$\therefore dP/dr \sim \frac{P_{surface} - P_{center}}{R_{surface} - 0} = \frac{-P_c}{R_\odot} = -\frac{G M_\odot \bar{\rho}_\odot}{R_\odot^2}$$

$$\therefore P_c \approx \frac{G M_\odot \bar{\rho}_\odot}{R_\odot} = \frac{6.7\times10^{-8}\times 2\times10^{33}\times 1.4}{6.96\times10^{10}}$$

$$\sim 2.8\times10^{15} \text{ dynes cm}^{-2} \quad (2.7\times10^{14}\,\text{N m}^{-2})$$

BUT M_r and $\rho_r \equiv \rho(r)$ VARY WITH RADIUS

REALLY $$P_c = \int_{P_s}^{P_c} dP = -\int_{R_\odot}^{R_c} \frac{G M_r \rho}{r^2}\,dr$$

$$\Rightarrow 2.34\times10^{17} \quad \text{OR} \quad 2.34\times10^{16}\,\text{N m}^{-2}$$

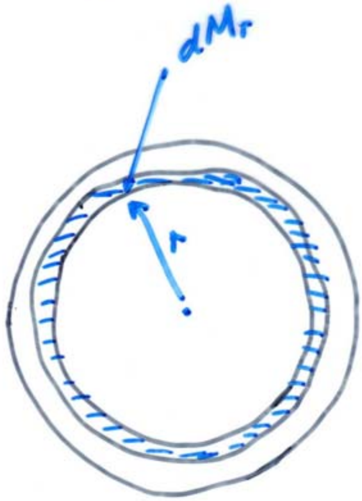
since ρ much greater than $\bar{\rho}$ near center

1 atm of pressure $= 1\times10^5$ N m^{-2} $\rightarrow P_c \sim 2.3\times10^{11}$ atm

TO IMPROVE CALCULATION NEED TO KNOW

ALSO **VARIATION OF M_r WITH RADIUS** (and ρ_r)

MASS CONSERVATION EQUATION



SPHERICALLY SYMMETRIC STAR

CONSIDER A SHELL MASS dM_r, THICKNESS dr AT DISTANCE r FROM CENTER OF STAR

$$dr \ll r$$

$\therefore$ shell volume

$$dV \simeq 4\pi r^2 dr$$

for <u>local</u> density ρ,

shell mass $dM_r = \rho (4\pi r^2 dr)$

$$\boxed{\therefore \frac{dM_r}{dr} = 4\pi r^2 \rho}$$

MASS CONSERVATION (CONTINUITY) EQUATION

$\Rightarrow$ VARIATION OF <u>INTERIOR MASS</u> WITH DISTANCE FROM CENTER

ORIGIN OF PRESSURE

NEED PRESSURE EQUATION OF STATE TO RELATE PRESSURE TO OTHER PROPERTIES OF STELLAR MATERIAL :

IDEAL GAS LAW = EXAMPLE OF EQUN OF STATE

$$PV = NkT$$

pressure volume # particles Boltzmann temperature

→ dependence of pressure on other properties

IN ASTROPHYSICAL SITUATIONS NOT ALWAYS APPLICABLE: USE PRESSURE INTEGRAL

$$\boxed{P = \frac{1}{3}\int_0^\infty n_p\, p\, v\, dp}$$

where $n_p\,dp$ = number of particles/unit volume (distribution fn) with momenta between p and dp

$$= \frac{N_p}{\Delta V}\,dp$$

($\frac{1}{3}$ since for particles in random motion probability of motion in x, y, z directions equal)

Pressure integral gives P, if distribution function $n_p dp$ known

Applies to massive and massless (photons) particles

For massive particles, $p = mv \rightarrow P = \frac{1}{3}\int_0^\infty m n_v v^2\, dv$

For ideal gas $n_v\, dv$ from Maxwellian:

$$n_v\, dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2\, dv$$

pressure integral $\Rightarrow P_g = nkT = \dfrac{NkT}{V}$ ideal gas law

IN ASTROPHYSICS, DIFFERENT KINDS OF PARTICLES (GAS ATOMS)

$\therefore$ let particle number density $n \left(\equiv N/V\right)$

$= \frac{\rho}{\bar{m}}$, where $\bar{m}$ is average particle mass

$$\therefore P_g = nkT = \rho \frac{kT}{\bar{m}}$$

DEFINE MEAN MOLECULAR WEIGHT $\mu \equiv \frac{\bar{m}}{m_H}$

= average mass of gas particle in units of m_H

$m_H = 1.67 \times 10^{-24}$ gms, 1.67×10^{-27} kg

$$\therefore \boxed{P_g = \frac{\rho kT}{\mu m_H}}$$

AN ASIDE

COMBINING IDEAL GAS LAW & PRESSURE INTEGRAL → AVERAGE K.E./PTCLE

$$P_g = nkT = \frac{1}{3}\int_0^\infty m n_v v^2 dv$$

$$\therefore \frac{1}{n}\int_0^\infty n_v v^2 dv = \frac{3kT}{m}$$

$\parallel$

integral average of v^2 weighted by Maxwell-Boltzmann distribution function

$$\therefore \bar{v}^2 = \frac{3kT}{m}$$

$$\therefore \frac{1}{2} m \bar{v}^2 = \frac{3}{2} kT$$

recall: factor of $\frac{1}{3}$ because 3 degrees of freedom for randomly moving particle

$= \frac{1}{2} kT$ per degree of freedom

RADIATION PRESSURE ALSO CONTRIBUTES: *massless particles*

FROM RELATIVISTIC ENERGY EQUATION $E^2 = p^2c^2 + m^2c^4$

where p = momentum & mc^2 = rest energy

$\therefore$ FOR PHOTON, MASS = 0 $\therefore E = pc = h\nu$

& $v = c$ and $p = h\nu/c$

PRESSURE INTEGRAL $P_{rad} = \frac{1}{3}\int_0^\infty n_p p v \, dp$

$\therefore$ using identity $n_p dp = n_\nu d\nu$, $P_{rad} = \frac{1}{3}\int_0^\infty n_\nu \cdot \frac{h\nu}{c} \cdot c \, d\nu$

$\therefore P_{rad} = \frac{1}{3}\int_0^\infty h\nu \, n_\nu \, d\nu$ *$n_\nu d\nu$ = number density of photons with frequency between ν and $\nu + d\nu$*

And ENERGY DENSITY $U_\nu d\nu = h\nu n_\nu d\nu$

$\therefore P_{rad} = \frac{1}{3}\int_0^\infty U_\nu \, d\nu = \frac{1}{3}\int_0^\infty \frac{4\pi}{c} B_\nu \, d\nu$

$\therefore P_{rad} = \frac{1}{3}\int_0^\infty \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1}$

$\int_0^\infty \frac{u^3 du}{e^u - 1} = \frac{\pi^4}{15}$ $\therefore$ let $u = h\nu/kT$, $\nu = \frac{ukT}{h}$, $d\nu = \frac{du \, kT}{h}$

$\therefore P_{rad} = \frac{1}{3}\int_0^\infty \frac{8\pi h}{c^3} \frac{k^3T^3}{h^3} \frac{u^3}{e^u - 1} \cdot \frac{kT}{h} du = \frac{8\pi}{3c^3} \frac{k^4T^4}{h^3} \frac{\pi^4}{15} = \frac{8\pi^5}{3c^3} \frac{k^4T^4}{15h^3}$

Stefan-Boltzmann constant $\sigma = \frac{2\pi^5 k^4}{15c^2h^3}$

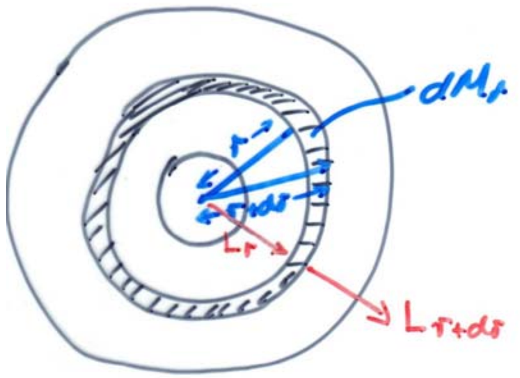
$\therefore P_{rad} = \frac{4}{3}\frac{\sigma}{c}T^4 = \frac{1}{3}aT^4$, a = radiation constant

$\therefore P_{total} = \frac{\rho k T}{\mu m_H} + \frac{1}{3}aT^4$ $\left(= P_{gas} + P_{radiation}\right)$

RATE OF ENERGY OUTPUT BY STAR

= LUMINOSITY

- GOVERNED BY <u>ENERGY CONSERVATION EQUATION</u>



spherical shell

L_r = energy flux

= interior luminosity due to energy generated interior to radius r

= energy passing through surface at r per unit time

Let $\mathcal{E}$ be amount of energy released in star / unit time / unit mass

$\therefore dL_r = L_{r+dr} - L_r = \mathcal{E}\, dM_r = 4\pi r^2 \rho \mathcal{E}\, dr$

for conservation of energy

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho \mathcal{E}}$$ - LUMINOSITY GRADIENT EQUATION

where $\mathcal{E}$ increases towards center

$\equiv$ ENERGY CONSERVATION EQU$^{\underline{N}}$

SOURCE OF ENERGY? GRAVITATIONAL COLLAPSE?

from continuity (mass conservation) equation

$$M_r \sim \frac{4}{3}\pi r^3 \bar{\rho}$$

gravitational P.E. $U_g = -\frac{G M m}{r^2}$

& $dU_{g_i} = -\frac{G M_r 4\pi r^2 \rho \, dr}{r}$ since $dm = 4\pi r^2 \rho \, dr$ (for shell dm, dr)

$\therefore$ integrating over all shells

$$U_g = -4\pi G \int_0^R M_r \bar{\rho} \, r \, dr$$

$$\therefore U_g = -\frac{16\pi^2}{3} G \bar{\rho}^2 \int_0^R r^4 \, dr \sim -\frac{16\pi^2}{15} G \bar{\rho}^2 R^5$$

$$= -\frac{3}{5} \frac{G M^2}{R}$$ since $\bar{\rho}^2 = \frac{9M^2}{16\pi^2 R^6}$

$U_g = -\frac{3}{5} \frac{G M^2}{R}$ and $E \sim -\frac{3}{10} \frac{G M^2}{R}$

from virial theorem

only 50% of grav^l pot^l energy available for release

COULD SOLAR ENERGY HAVE BEEN RESULT OF COLLAPSE FROM MUCH LARGER STAR?

Let R_i = initial radius $\quad R_i \gg R_\odot$

& ENERGY GENERATED BY COLLAPSE = ΔE_g

$$\Delta E_g = -(E_f - E_i) \simeq -E_f$$

$$\simeq \frac{3}{10} \frac{G M_\odot^2}{R_\odot}$$

$$\therefore \Delta E = \frac{3 \times 6.67 \times 10^{-8} \times 4 \times 10^{66}}{10 \times 6.96 \times 10^{10}}$$

$$\sim 1.1 \times 10^{48} \text{ ergs/sec}$$

$$L_\odot \sim 3.9 \times 10^{33} \text{ ergs/sec}$$

$\therefore$ SUN COULD 'SHINE' FOR $\dfrac{1.1 \times 10^{48}}{4 \times 10^{33} \times 3 \times 10^{7}}$ yrs

$= 10^7$ YRS

MOON ROCKS → AGE OF SUN $\sim 4 \times 10^9$ YRS

Note $\quad \Delta E_g / L_\odot = t_{KH}$ = KELVIN-HELMHOLTZ TIME SCALE

$\equiv$ collapse time

- GRAVITATIONAL PROCESSES INSUFFICIENT TO FUEL SUN/STARS? YES

- CHEMICAL PROCESSES?

SINCE BASED ON INTERACTIONS OF ORBITAL ELECTRONS → FEW eV PER ATOM
BUT MASS OF SUN ⇒ TOO FEW ATOMS TO SUSTAIN LUMINOSITY

- NUCLEAR PROCESSES?
  - → TRANSFORMATION OF NUCLEI
    - HYDROGEN 'BURNING'
    - HELIUM 'BURNING'
  - → MeV INTERACTIONS

NOMENCLATURE: NUCLEONS = PROTONS AND NEUTRONS

Z = NUMBER OF PROTONS N = NUMBER OF NEUTRONS

for NEUTRAL ATOMS: Z = number of electrons

ISOTOPES: SAME Z FOR EACH ELEMENT
DIFFERENT N

TOTAL NUMBER OF NUCLEONS IN GIVEN ISOTOPE

$= A = Z + N =$ mass number electron mass small

$m_p = 1.67 \times 10^{-24}$ gm, $m_N = 1.67 \times 10^{-24}$ gm

$m_e = 9.1 \times 10^{-28}$ gm

OFTEN EXPRESSED IN TERMS OF ATOMIC MASS UNIT

$$u = \tfrac{1}{12} \text{ mass of } C^{12} \sim 1.66 \times 10^{-27} kg$$

$$\Rightarrow \quad m_p = 1.67.. \times 10^{-27} kg = 1.007u$$

$$m_n = 1.67.. \times 10^{-27} kg = 1.009u$$

$$m_e = 0.0005u$$

OR IN TERMS OF REST MASS ENERGY $E = mc^2$

$$m = E/c^2 \; kg = E/c^2(1.66 \times 10^{-24}) \; u \leftarrow \text{cgs}$$

$$\therefore 1 \; u = \frac{1.66 \times 10^{-24} \times 9 \times 10^{20}}{1.6 \times 10^{-12} c^2} eV$$

$$\therefore 1u = \frac{1.494}{1.6} \times \frac{10^9}{c^2} = 931.49 \; MeV/c^2$$

Usually mass expressed in terms of rest mass energy with c^2 implicit

CONSIDER SIMPLEST ISOTOPE HYDROGEN

1 proton 1 electron

mass $m_H = 1.007825u... < m_p + m_e$

$$\begin{array}{r} 1.00727646... \\ +0.00054858... \\ \hline 1.0078250... \end{array}$$

Thus $\Rightarrow 1.00782503214u^{+}... = 1.00782504679u$

$$\Delta m = 1465 \times 10^{-8} \times 931.49 \times 10^6$$

$$= 1.465 \times 9.3149 = 13.6 eV$$

$= \chi_i$ IONIZATION POTENTIAL (GROUND STATE)

· ENERGY ($\equiv$ MASS) LOST WHEN PROTON - ELECTRON COMBINE (FUSION) = BINDING ENERGY

BINDING ENERGY → TOTAL ENERGY RELEASED IN FORMING NUCLEUS

CONSIDER HELIUM: 2 PROTONS 2 NEUTRONS

$m_{He} = 4.002603\,u$

$4 \times m_H = 4.03130013\,u$ neglecting neutrinos low mass remnants

(4 H nuclei → 1 He nucleus + remnants)

$\therefore \Delta m$ = binding energy = $0.028697\,u$

$\simeq 0.7\%$ of H mass ($\sim 26.7\,eV$)

$\therefore$ Energy released = 0.7% of 4 H's mass

SUFFICIENT ENERGY TO POWER SUN FOR $> 10^9$ YRS?

SUPPOSE: SUN 100% HYDROGEN

SUPPOSE: HYDROGEN BURNING (TO FORM HE) POSSIBLE ONLY IN INNERMOST 10% OF MASS

and 0.7% OF H-MASS WILL BE RELEASED

$$\therefore E_{nuclear} = Mc^2 = M_\odot \times 0.1 \times 0.007 \times c^2$$

$$= 2 \times 10^{33} \times 7 \times 10^{-4} \times 9 \times 10^{20}$$

$$\sim 1.3 \times 10^{51} \text{ ergs}$$

THIS Energy RELEASED AT RATE $L_\odot$ OVER time

$$t_{nuclear} = \frac{E}{L_\odot} = \frac{1.3 \times 10^{51}}{4 \times 10^{33} \times 3 \times 10^7} \text{ years}$$

$$= \frac{1.3}{1.2} \times 10^{10}$$

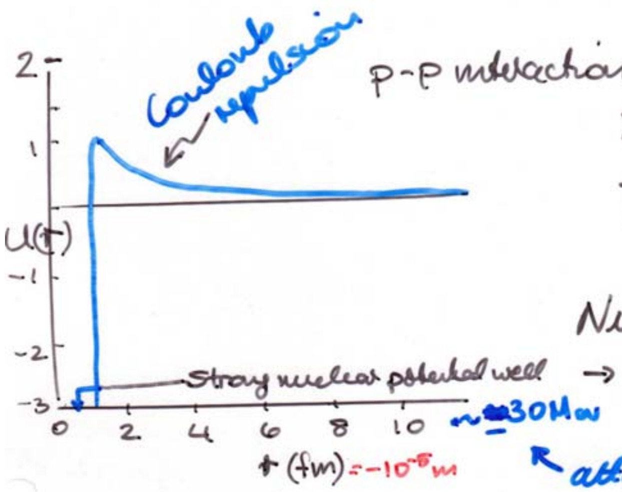
$$\therefore t_{nuclear} \sim 10^{10} \text{ years}$$

COMPATIBLE WITH AGE OF MOON ROCKS!

IF NUCLEAR REACTIONS - FUSION - ARE POSSIBLE

= SOURCE OF STELLAR ENERGY

CAN THEY OCCUR IN STELLAR INTERIORS?



Potential energy as function of separation, r, between nuclei

Nuclei +vely charged → Coulomb potential energy barrier outside nucleus

Assume thermal energy provides for overcoming Coulomb barrier

$\therefore \frac{1}{2}\mu_m \bar{v}^2 = \frac{3}{2}kT_{classical} = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{r}$ $Z_1 Z_2$ # protons in each nucleus

∴ TEMPERATURE TO OVERCOME COULOMB BARRIER

$= T_{classical} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 r} \sim 10^{10}K$ for 2 protons $Z_1 = Z_2$

BUT $T_{c\odot} = 1.57 \times 10^7 K$ = central temp Sun = TOO LOW

HEISENBERG $\Delta x \Delta p_x > \hbar/2$ - can't know both precisely

∴ COULD HAVE ONE PARTICLE IN POT^L WELL OF OTHER

≡ TUNNELLING

$T_{QUANTUM} \sim 10^7 K$ → nuclear reactions possible